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(Residential Autonomous College under University of Calcutta)

B.A./B.Sc. SIXTH SEMESTER EXAMINATION, MAY 2014

THIRD YEAR

MATHEMATICS (Honours)

Paper : VII

Date : 19/05/2014

Time : 11 am – 3 pm

Full Marks : 100

[Use a separate Answer book for each Group]

Group – A

Answer **any five** questions : [5×10]

1. a) Let $f:[0,1]\to\mathbb{R}$ be defined as follows

$$f(x)=\begin{cases} x\cos\dfrac{\pi}{2x}, & x\neq 0\\ 0, & x=0\end{cases}$$

Check whether f is of BV over [0,1]

b) If $f:[a,b]\to\mathbb{R}$ be Riemann integrable, prove that $F(x)=\int_a^x f(t)\,dt$ is of bounded variation over [a,b].

c) A function f is continuous on $\mathbb{R}$ and $\int_{-x}^{x} f(t)\,dt=0\ \forall x\in\mathbb{R}$. Prove that f is an odd function on $\mathbb{R}$. [4+3+3]

2. a) The functions $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are both continuous on [a,b] and $\int_a^b |f-g|=0$. Prove that f = g.

b) A function f is defined on [0,1] by f(0) = 0 and

f(x) = 0 if x is irrational

$=\frac{1}{q}$ if $x=\frac{p}{q}$ where p, q are positive integers prime to each other

Show that f is integrable on [0,1].

c) Prove that for $0<a<b<+\infty$, $\left|\int_a^b \sin(x^2)\,dx\right|\le\frac{1}{a}$ [3+3+4]

3. a) Let a function $f:[a,b]\to\mathbb{R}$ be integrable on [a,b] and $f(x)\ge 0\ \forall x\in[a,b]$. Let $\exists$ a point $c\in[a,b]$ such that f is continuous at c and $f(c)>0$, prove that $\int_a^b f>0$.

b) If f is a monotone decreasing function on $[1,\infty)$ and $f(x)>0\ \forall x\in[1,\infty)$, then the improper integral $\int_1^\infty f(x)\,dx$ and the infinite series $\sum_1^\infty f(n)$ converge or diverge together. [5+5]

4. a) Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if m, n are both positive.

b) Prove that $\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$.

c) Show that the following improper integral is absolutely convergent : $\int_0^\infty e^{-ax}\cos bx\,dx$. [3+4+3]

5. a) Change the order of integration $\int_{1/3}^{2/3} dx\int_{x^2}^{\sqrt{x}} f(x,y)\,dy$ where f(x,y) is continuous function of x and y.

b) Express $\iint_S f(x,y)\,dx\,dy$ interms of iterated integrals where S is the trapezoid with the vertices (1,1), (5,1), (4,4) and (2,4). Here f(x,y) is continuous function of x and y.

c) Let $f(x,y)=\begin{cases}\frac{1}{2}, \text{when } x \text{ is rational}\\ y, \text{ when } x \text{ is irational}\end{cases}$

Test the existence of $\int_0^1\left\{\int_0^1 f\,dy\right\}dx$ and $\int_0^1\left\{\int_0^1 f\,dx\right\}dy$. [3+3+4]

6. a) If $I=\int_0^a e^{-x^2}dx$, prove by expressing I as a double integral, that $\frac{\pi}{4}(1-e^{-a^2})\le I^2\le\frac{\pi}{4}\left(1-e^{-2a^2}\right)$.

b) Evaluate **any one** of the following :

i) $\iint_R \sqrt{4a^2-x^2-y^2}\,dxdy$ where R is the upper half of the circle $x^2+y^2-2ax=0$.

ii) $\iiint_E \frac{dxdydz}{x^2+y^2+(z-2)^2}$ where E is the sphere $x^2+y^2+z^2\le 1$. [5+5]

7. a) Let a function f(x) of period 2π can be expanded in a trigonometric series of form $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ (all a_n's, b_n's are constants), which converges uniformly to f(x) in $[-\pi,\pi]$. Show that this series is the Fourier series of f(x).

b) Find a Fourier series representing f(x) on $-\pi<x<\pi$ where

$$f(x) = \begin{cases} 0, & -\pi<x\le 0\\ \frac{\pi x}{4}, & 0<x<\pi\end{cases}$$

and deduce that $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$. [5+5]

8. a) Using the Fourier series expansion of $f(x)=|x|$ in $[-\pi,\pi]$, show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}$.

b) Correct or justify the statement : $\sum_{n=1}^{\infty}\frac{\cos nx}{\sqrt{n}}$ is not Fourier series of any Riemann-integrable function in $[-\pi,\pi]$.

c) Given $x(\pi-x)=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3}$, $0\le x\le\pi$ deduce that $x=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$, $0\le x\le\pi$. [4+3+3]

Group – B

Unit - I

Answer **any three** questions : [3×10]

9. a) The chance that a doctor will diagnose a certain disease correctly in 60%. The chance that the patient will die after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of the doctor having the disease dies. What is the probability that the disease was diagnosed correctly?

b) For n events A_1, A_2 ..., A_n connected with random experiment E, show that $P(A_1A_2\ldots A_n)\ge\sum_{i=1}^{n}P(A_i)-(n-1)$.

c) A and B throw a pair of fair dice. A wins, if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6.If A begins, what is the chance of B's winning? [4+3+3]

10. a) Find the value of k so that the function f_x given by $f_x(x) = \begin{cases} kx(2-x) & , \quad 0<x<2 \\ 0 & , \quad \text{elsewhere} \end{cases}$

is a probability density function. Construct the distribution function and compute P(X>1).

b) Prove that in n-Bernoulli trials with probability of failure q, the probability of at most k success is $\dfrac{\int_0^q x^{n-k-1}(1-x)^k dx}{\int_0^1 x^{n-k-1}(1-x)^k dx}$. [5+5]

11. a) Two people agree to meet at a definite place between 12 and 1 o'clock with the understanding that each will wait 20 minutes for the other. What is the probability that they will meet?

b) Two numbers are chosen at random between 0 and 2, find the probability that their sum of squares is less than 2.

c) If X is a binomial (n,p) variate, then prove that $\mu_{k+1} = p(1-p)\left(\eta k\mu_{k-1} + \dfrac{d\mu_k}{dp}\right)$, where μ_k is the kth central moment. [4+3+3]

12. a) Prove that the moment generating function of uniform distribution over $(-a,a)$ is $\dfrac{\sin h\, at}{at}$. Hence find k-th central moment.

b) If X,Y are two independent random variables, then prove that they are uncorrelated but not conversely.

c) If the regression lines are $x-2y+1=0$ & $2x-3y+1=0$, find the mean of X, the mean of Y and the correlation coefficient between X & Y. [4+3+3]

13. a) A random variable X has the probability density function f_x given by $f_x(x) = \begin{cases} 12x^2(1-x) & , \quad 0<x<1 \\ 0 & , \quad \text{elsewhere} \end{cases}$

Compute $P(1\times - m_x \,|\geq 2\sigma_x)$ and compare it with the limit given by Tchebycheff's inequility, where m_x and σ_x are respectively the mean and standard deviation of X.

b) An unbiased die is thrown 1200 times. Find the minimum value of the probability of getting 160 to 240 sixes.

c) Is there any distribution of a random variable X such that $P(1X - m_x \,|\leq 2\sigma_x) = 0\cdot 718$? [4+3+3]

Unit - II

Answer **any two** questions : [2×10]

14. a) Let $f: G \to \mathbb{C}$ where $G(\subset \mathbb{C})$ be a region. Let $f(z) = u(x,y) + iv(x,y)$ and $z_0 \in G$, $z_0 = x_0 + iy_0$ where $x_0, y_0 \in \mathbb{R}$.

Let u, v, u_x, v_x, u_y, v_y be continuous at (x_0, y_0) and u,v satisfy Cauchy-Riemann equations at (x_0,y_0). Prove that f is differentiable at z_0.

b) Justify the statement : sin z is entire but unbounded function in $\mathbb{C}$.

c) Check $f(z) = \dfrac{xy^2(x+iy)}{x^4+y^4}$, satisfies Cauchy-Riemann equation at z = 0 or not, $z = (x+iy), x, y \in \mathbb{R}$. [4+3+3]

15. a) If $G(\subset \mathbb{C})$ be a region and $f: G \to \mathbb{C}$ is analytic in G with $f'(z) = 0$ for all z in G, prove that f is a constant.

b) Show that Logz is not continuous on the negative real axis, (Here Logz denotes the principal part of Logarithmic function)

c) Let p(x,y) and q(x,y) are harmonic functions in domain D. Show that $p_y - q_x + i(p_x + q_y)$ is analytic in D. [4+3+3]

16. a) Prove that a Power series and its K-times derived power series both have the same radius of Convergence.

b) Find the radius of Convergence of the power series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{z^n + 1}$ and prove that $(2-z)f(z) - 2 \to 0$ as $z \to 0$.

c) Let $f(z) = u + iv$ be an analytic function. If $u - v = (x-y)(x^2 + 4xy + y^2)$, find f(z) in terms of z. [3+3+4]